

 $N = \{1, 2, 3\}, v(\{i\}) = 0 \text{ for } i \in \{1, 2, 3\}$

 $v(\{1,2\})=5,\,v(\{1,3\})=6,\,v(\{2,3\})=6$

v(N)=8

Which payoff should we prefer? *x* or *y*? Let us write the excess in the decreasing order (from the greatest excess to

Some properties of \leq_{lex} and its strict version

 $y = \langle 2, 3, 3 \rangle$

coalition $\mathcal{C} = e(\mathcal{C}, y)$

-2 -3 -3

0

1

0

0

Lecture 5: The nucleolus 4

{1}

{2}

{3}

{1.2}

{1,3}

{2,3}

{1,2,3}

(1,0,0,0,-2,-3,-3)

Let us consider two payoff vectors $x = \langle 3, 3, 2 \rangle$ and $y = \langle 2, 3, 3 \rangle$. Let e(x) denote the sequence of **excesses** of all coalitions at *x*.

= (3,3,2)

-3

-3 -2

-1

1

1

0

coalition $\mathcal{C} = e(\mathcal{C}, x)$

{2}

{3}

{1,2}

{1,3}

{2,3}

{1,2,3}

(1, 1, 0, -1, -2, -3, -3)

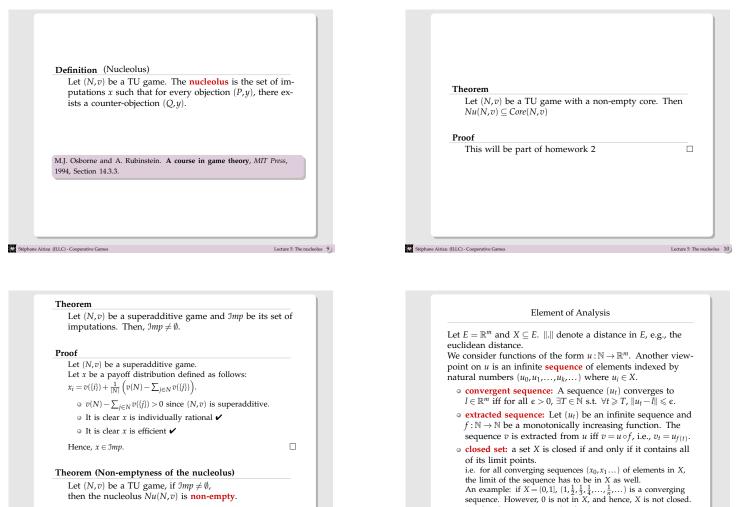
the smallest)

Stéphane Airiau (ILLC) - Cooperative Game

Let (N, v) be a TU game. Objections are made by coalitions instead of individual agents. Let $P \subseteq N$ be a coalition that expresses an objection.
A pair (P, y) , in which $P \subseteq N$ and y is an imputation, is an objection to x iff $e(P, x) > e(P, y)$.
Our excess for coalition P is too large at x, payoff y reduces it.
A coalition (Q,y) is a counter-objection to the objection (P,y) when $e(Q,y) > e(Q,x)$ and $e(Q,y) \ge e(P,x)$.
Our excess under y is larger than it was under x for coalition Q! Furthermore, our excess at y is larger than what your excess was at x!
An imputation fails to be stable if the excess of some coalition P can be reduced without increasing the excess of some other coalition to a level at least as large as that of the original excess of P .

Definition (Nucleolus) Let (N, v) be a TU game.

Let $\Im mp$ be the set of all imputations. The **nucleolus** Nu(N, v) is the set $Nu(N, v) = \{x \in \Im mp \mid \forall y \in \Im mp \ e(y)^{\blacktriangleright} \ge_{lex} e(x)^{\blacktriangleright}\}$



then the nucleolus Nu(N, v) is **non-empty**.

Stéphane Airiau (ILLC) - Cooperative Games

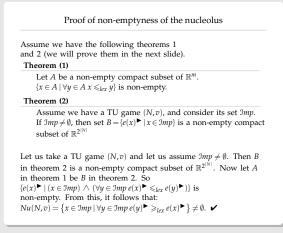
Element	~f	A	
Element	OI.	Ana	VSIS

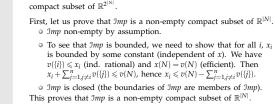
- **bounded set:** A subset $X \subseteq \mathbb{R}^m$ is **bounded** if it is contained in a ball of finite radius, i.e. $\exists c \in \mathbb{R}^m$ and $\exists r \in \mathbb{R}^+ \text{ s.t. } \forall x \in X ||x - c|| \leq r.$
- **compact set:** A subset $X \subseteq \mathbb{R}^m$ is a **compact** set iff from all sequences in X, we can extract a convergent sequence in X.
- A set is **compact** set of \mathbb{R}^m iff it is **closed** and **bounded**.
- **convex set:** A set *X* is convex iff $\forall (x,y) \in X^2$, $\forall \alpha \in [0,1]$, $\alpha x + (1 - \alpha)y \in X$ (i.e. all points in a line from x to y is contained in X).
- **continuous function:** Let $X \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$. *f* is **continuous** at $x_0 \in X$ iff $\forall \epsilon \in \mathbb{R}$, $\epsilon > 0$, $\exists \delta \in \mathbb{R}$, $\delta > 0$ s.t. $\forall x \in X$ s.t. $||x - x_0|| < \delta$, we have $||f(x) - f(x_o)|| < \epsilon$, i.e. $\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in X \quad ||x - x_0|| < \delta \Rightarrow ||f(x) - f(x_0)|| < \epsilon.$

Stéphane Airiau (ILLC) - Cooperative Games

Lecture 5: The nucleolus 13)

Lecture 5: The nucleolus 11)





"A closed set contains its borders".

Element of Analysis

Thm A₁ If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $X \subseteq E$ is a non-empty

then f(X) is a non-empty compact subset of \mathbb{R}^m .

Thm A₂ Extreme value theorem: Let X be a non-empty compact

Then f is bounded and it reaches its supremum.

Thm A₃ Let X be a non-empty compact subset of \mathbb{R}^n . $f: X \to \mathbb{R}$ is

continuous iff for every closed subset $B \subseteq \mathbb{R}$, the set

Proof of theorem 2 Let (N, v) be a TU game and consider its set $\exists mp$. Let us assume

that $\Im mp \neq \emptyset$ to prove that $B = \{e(x)^{\blacktriangleright} \mid x \in \Im mp\}$ is a non-empty

subset of \mathbb{R}^n , $f: X \to \mathbb{R}$ a **continuous** function.

Stéphane Airiau (ILLC) - Cooperative Games

Let $X \subseteq \mathbb{R}^n$.

Stéphane Airiau (ILLC) - Cooperative Game

compact subset of \mathbb{R}^n ,

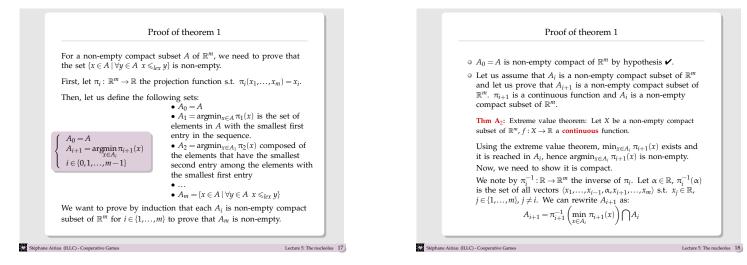
 $f^{-1}(B)$ is compact.

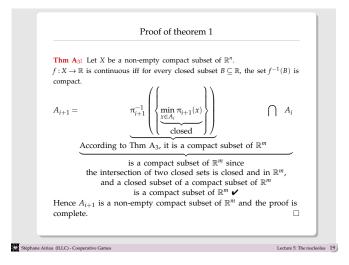
Thm A₁ If $f: E \to \mathbb{R}^m$ is continuous, $X \subseteq E$ is a non-empty compact subset of $\mathbb{R}^n,$ then f(X) is a non-empty compact subset of \mathbb{R}^m

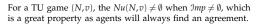
 $e()^{\blacktriangleright}$ is a continuous function and $\Im mp$ is a non-empty and compact subset of $\mathbb{R}^{2^{[N]}}$. Using thm A₁, $e(\mathfrak{I}mp)^{\bigstar} = \{e(x)^{\bigstar} | x \in \mathfrak{I}mp\}$ is a non-empty compact subset of $\mathbb{R}^{2^{[N]}}$.

Lecture 5: The nucleolus 14

Lecture 5: The nucleolus 12







Theorem

The nucleolus has at most one element

In other words, there is **one** agreement which is stable according to the nucleolus.

To prove this, we need theorems 3 and 4.

Theorem (3)

Let *A* be a non-empty convex subset of \mathbb{R}^m Then the set $\{x \in A \mid \forall y \in A \ x^{\blacktriangleright} \leq_{lex} y^{\blacktriangleright}\}$ has at most one element.

Theorem (4)

nal and efficient.

vidually rational.

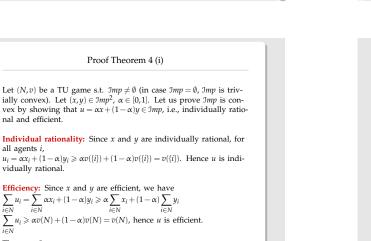
Thus, $u \in \Im mp$

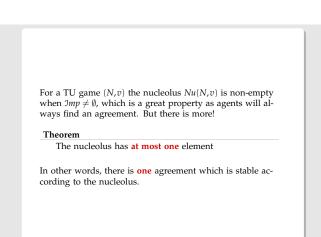
all agents i,

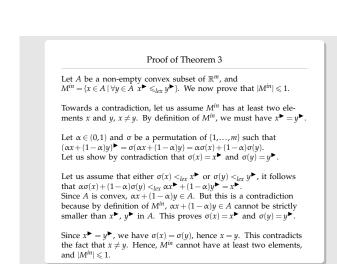
Let (N, v) be a TU game such that $\Im mp \neq \emptyset$. (i) $\ensuremath{\mathbb{J}mp}$ is a non-empty and convex subset of $\ensuremath{\mathbb{R}}^{|N|}$ (ii) $\{e(x) \mid x \in \Im mp\}$ is a non-empty convex subset of $\mathbb{R}^{2^{|N|}}$

iphane Airiau (ILLC) - Cooperative Games STA St

Lecture 5: The nucleolus 21)







Stéphane Airiau (ILLC) - Cooperative Game

Stéphane Airiau (ILLC) - Cooperative Games

Lecture 5: The nucle lus 22

Lecture 5: The nucleolus 20

Proof Theorem 4 (ii)

Let (N, v) be a TU game and $\exists mp$ its set of imputations. We need to show $\{e(z) \mid z \in \exists mp\}$ is a non-empty convex subset of \mathbb{R}^m . Let $(x,y) \in \exists mp^2$, $\alpha \in [0,1]$, and $\mathbb{C} \subseteq N$ and we consider the sequence $\alpha e(x) + (1 - \alpha)e(y)$, and we look at the entry corresponding to coali-tion e^{α} . tion C.

 $\left(\alpha e(x)+(1-\alpha)e(y)\right)_{\mathfrak{C}} \quad = \quad \alpha e(\mathfrak{C},x)+(1-\alpha)e(\mathfrak{C},y)$

- $= \alpha(v(\mathcal{C}) x(\mathcal{C})) + (1 \alpha)(v(\mathcal{C}) y(\mathcal{C}))$
- $= v(\mathcal{C}) (\alpha x(\mathcal{C}) + (1 \alpha)y(\mathcal{C}))$
- $= v(\mathfrak{C}) ([\alpha x + (1 \alpha)y](\mathfrak{C}))$
- $= e(\alpha x + (1 \alpha)y, \mathcal{C})$

Since the previous equality is valid for all $\mathcal{C}\subseteq N,$ both sequences are equal: $\alpha e(x) + (1 - \alpha)e(y) = e(\alpha x + (1 - \alpha)y).$

Since $\exists mp$ is convex, $\alpha x + (1 - \alpha)y \in \exists mp$, it follows that $e(\alpha x + (1 - \alpha)y) \in \{e(z) \mid z \in \exists mp\}$. Hence, $\{e(z) \mid z \in \exists mp\}$ is convex.



Let (N, v) be a TU game, and $\Im mp$ its set of imputations. **Theorem 4(ii):** $\{e(x) \mid x \in \Im mp\}$ is a non-empty convex subset of $\mathbb{R}^{2^{|N|}}$.

Theorem 3: If *A* is a non-empty convex subset of \mathbb{R}^m , then the set $\{x \in A \mid \forall y \in A \ x^{\blacktriangleright} \leq_{lex} y^{\blacktriangleright}\}$ has at most one element.

Applying theorem 3 with $A = \{e(x) \mid x \in \exists mp\}$ we obtain $B = \{e(x) \mid x \in \exists mp \land \forall y \in \exists mp \ e(x)^{\blacktriangleright} \leq_{lex} e(y)^{\blacktriangleright}\}$ has at most on element.

B is the image of the nucleolus under the function *e*. We need to make sure that an e(x) corresponds to at most one element in $\exists mp$. This is true since for $(x, y) \in \exists mp^2$, we have $x \neq y \Rightarrow e(x) \neq e(y)$.

Hence $Nu(N, v) = \{x \mid x \in \exists mp \land \forall y \in \exists mp \ e(x)^{\blacktriangleright} \leq_{lex} e(y)^{\blacktriangleright}\}$ has at most one element!

Stéphane Airiau (ILLC) - Cooperative Games

need to nt in <i>Jmp.</i> <i>e</i> (<i>y</i>). } has at	 coalitions and the lexicographic ordering to any two imputations. We defined the nucleolus for a TU game. pros: If the set of imputations is non-empty, the non-empty. The nucleolus contains at most one eleme When the core is non-empty, the nucleolu the core. cons: Difficult to compute. 	e nucleolus is nt.
Lecture 5: The nucleolus 25	Stéphane Airiau (ILLC) - Cooperative Games	Lecture 5: 1
Lecture X ine nucleonits 23	Suprane Annai (ILL)- Corperative Cames	Lecture 5: 1

Summary

• We defined the excess of a coalition at a payoff distribution, which can model the complaints of the

 ${\scriptstyle \circ}$ We used the ordered sequence of excesses over all

Lecture 5: The nucleolus 26

members in a coalition.

Coming next
• The kernel , also a member of the bargaining set family, also based on the excess.

Stéphane Airiau (ILLC) - Cooperative Games

Lecture 5: The nucleolus 27